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ORLICZ PROPERTY AND COTYPE IN SYMMETRIC SEQUENCE SPACES

BY

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ABSTRACT

We construct a symmetric sequence space that satisfies the Orlicz property but that fails to be of cotype 2.

1. Introduction

A Banach space X is said to have the Orlicz property if there exists a number K such that given any $n > 0$ and any vectors $x_1, \ldots, x_n$ of X, one can find signs $\eta_i \in \{-1, 1\}$ such that

$$(1.1) \qquad \left(\sum_{i\leq n} \|x_i\|^2\right)^{1/2} \leq K \|\sum_{i\leq n} \eta_i x_i\|.$$

A Banach space is said to be of cotype 2 if there exists a number K such that given any n, and any vectors $x_1, \ldots, x_n$ of X, one has

$$(1.2) \qquad \left(\sum_{i\leq n} \|x_i\|^2\right)^{1/2} \leq K\, Av \|\sum_{i\leq n} \eta_i x_i\|$$

where Av denotes the average over all choices of signs $\eta_1, \ldots, \eta_n$. Since the average is less than the maximum, cotype 2 implies the Orlicz property. It had been

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open for some time whether the reverse implication holds. A counter example was constructed in [T]. The counter example is a Banach lattice. The aim of the present paper is to show that actually a counter example can be found that is a symmetric sequence space. That is, the space contains an unconditional basis $(e_i)_{i\geq 1}$, and the norm of a vector $x = \sum_{i\geq 1} x_i e_i$ is invariant under permutation of the coefficients.

THEOREM: *There exists a symmetric sequence space that satisfies the Orlicz property but fails to be of cotype 2.*

There is no doubt that there is much less room to construct the above example than the example of [T]. However, once the proper choice of parameters has been found, the proof is not more complicated. Several of the key ideas of the construction of [T] will be preserved in our construction, but the paper is written to be read independently of [T] and is self-contained.

2. The construction

Basic to the construction is a sequence (n_p) defined as follows. We take $n_{-1} = n_0 = 1/2$, and we define the sequence by induction using the following relation:

$$n_p = 2^{2p+4} n_{p-1} (2^{6p+2} n_{p-1}^2)^p. \tag{2.1}$$

(Thus, for $p \geq 1$, n_p is an integer.) There is nothing magic about this choice. Many other choices are possible. The reason for (2.1) will become clear later. To simplify notations, we set, for $p \geq 1$

(2.2) $$m_p = 2^{6p+2} n_{p-1}^2,$$

(2.3), $$k_p = 2^p n_{p-1},$$

(2.4). $$n'_p = 2k_p m_p^p.$$

Thus, we have

$$m_p = 2^{4p+4} k_p^2; \quad n_p = 2^{p+3} n'_p. \tag{2.5}$$

We observe that $k_1 = 1$ and we consider the function h on $\mathbb{N}$ such that for $p \geq 2$ we have

$$k_{p-1} \leq i < k_p \Rightarrow h(i) = \frac{2^p}{k_p}.$$

It follows that

$$\sum_{i\le k_p} h(i) \le \sum_{\ell\le p} 2^\ell \le 2^{p+1}. \tag{2.6}$$

Since $k_{p+1} > n_p$, we also have

$$\sum_{i\le n_p} h(i) \le 2^{p+2}. \tag{2.7}$$

We denote by H the set of functions of the type $h\circ\sigma$, where σ is any permutation of $\mathbb{N}$, and we now describe for $r\ge 1$ a class $\mathcal{F}_r$ of functions on $\mathbb{N}$. The class $\mathcal{F}_r$ consists of the functions that are positive and satisfy

$$\|f\|_\infty \le \frac{1}{n_{r-2}}, \tag{2.8}$$

and such that one can find for all $\ell\ge 0$, and all $j\le m_r^\ell$ functions $h_{\ell,j}\in H$, and numbers $\alpha_{\ell,j}$, $\alpha_{\ell,j}\ge 0$, $\sum_{\ell\ge 0, j\le m_r^\ell}\alpha_{\ell,j}\le 1$, such that

$$\forall i\in\mathbb{N},\quad f(i)\le \sum_{\ell\ge 0} 2^{-\ell}\sum_{j\le m_r^\ell}\alpha_{\ell,j}h_{\ell,j}(i). \tag{2.9}$$

We first prove two properties of $\mathcal{F}_r$, that will be crucial in establishing the Orlicz property.

LEMMA 2.1: *Consider functions $(f_s)_{s\le m_r}$ in $\mathcal{F}_r$. Consider numbers $\beta_s\ge 0$, $\sum_{s\le m_r}\beta_s = 1$. Then*

$$\frac{1}{2}\sum_{s\le m_r} f_s \in \mathcal{F}_r.$$

Proof: By (2.9), we know that (with obvious notations) for $s\le m_r$

$$f_s \le \sum_{\ell\ge 0} 2^{-\ell}\sum_{j\le m_r^\ell}\alpha_{\ell,j,s}h_{\ell,j,s},$$

so that

$$\frac{1}{2}\sum_{s\le m_r}\beta_s f_s \le \sum_{\ell\ge 0} 2^{-(\ell+1)}\sum_{\substack{j\le m_r^\ell\\ s\le m_r}}\beta_s\alpha_{\ell,j,s}h_{\ell,j,s}$$

and the point is that there are at most $m_r^{\ell+1}$ terms in the last summation above ∎

LEMMA 2.2: *Consider functions* $(f_s)_{s\le m_{r+1}}$ *in* $\mathcal{F}_r$. *Consider numbers* $\beta_s \ge 0$, $\sum_{s\le m_{r+1}} \beta_s = 1$. *Then one can find a set* A *of integers of cardinal at most* k_r, *such that the function defined by* $g(i) = 0$ *if* $i \in A$ *and* $g(i) = \frac{1}{2}\sum_{s\le m_{r+1}} \beta_s f_s(i)$ *if* $i \notin A$ *belongs to* $\mathcal{F}_{r+1}$.

Proof: Set $g' = \frac{1}{2}\sum_{s\le m_{r+1}} \beta_s f_s$. The fact that g' satisfies (2.9) (for $r+1$ rather than r) is proved as in Lemma 2.1, using the fact that $m_r^\ell m_{r+1} \le m_{r+1}^{\ell+1}$. The problem here is that g' may not satisfy $\|g'\|_\infty \le 1/n_{r-1}$. Consider a subset A of $\mathbb{N}$, of cardinality k_r. We observe that if $h' \in H$, we have

$$\sum_{i\in A} h'(i) \le \sum_{i\le k_r} h(i) \le 2^{r+1}$$

by (2.6). Thus we have by (2.3)

$$\sum_{i\in A} g'(i) \le 2^r = \frac{k_r}{n_{r-1}}.$$

It follows that if A has been chosen such that

$$i \in A, \quad j \notin A \Rightarrow g'(j) \le g'(i),$$

then $g'(j) \le 1/n_{r-1}$ for $j \notin A$. This concludes the proof. ∎

The third crucial property of $\mathcal{F}_r$ (that will ensure that X does not have cotype 2) is unfortunately more complex, and will be investigated in Section 3. We prepare that study with a simple fact.

LEMMA 2.3: *Consider* $f \in \mathcal{F}_r$. *Then we can write* $f = f_1 + f_2$, *where the following holds:*

(2.10) *The support of* f_1 *has cardinality* $\le n_r'$,

(2.11) $$\sum_{i\le n_r} f_2(i) \le 5.$$

Proof: Consider the functions $h_{\ell,j}$ as in (2.9). We observe that any function $h_{\ell,j}$ can be written $h_{\ell,j}^1 + h_{\ell,j}^2$, where $\|h_{\ell,j}^2\|_\infty \le 2^{r+1}/k_{r+1} = 1/n_r$ and where the support of $h_{\ell,j}^1$ has at most k_r elements. The function

$$g_1 = \sum_{0\le\ell\le r} 2^{-\ell} \sum_{j\le m_r^\ell} \alpha_{\ell,j} h_{\ell,j}^1$$

has a support of cardinality at most

$$k_r \sum_{\ell \le r} m_r^\ell \le 2k_r m_r^r = n_r'.$$

We set

$$g_2 = \sum_{\ell \le r} 2^{-\ell} \sum_{j \le m_r^\ell} \alpha_{\ell,j} h_{\ell,j}^2 + \sum_{\ell > r} 2^{-\ell} \sum_{j \le m_r^\ell} \alpha_{\ell,j} h_{\ell,j}.$$

We recall that

$$\|h_{\ell,j}^2\|_\infty \le 1/n_r \le 1/m_r^r$$

and that (as follows from (2.7)), for $h' \in H$, we have

$$\sum_{i \le n_r} h'(i) \le 2^{r+2}. \tag{2.12}$$

Thus

$$\begin{aligned}\sum_{i \le n_r} g_2(i) &\le \sum_{1 \le \ell \le r} 2^{-\ell} \sum_{j \le m_r^\ell} \alpha_{\ell,j} + \sum_{\ell > r} 2^{-\ell} 2^{r+2} \sum_{j \le m_r^\ell} \alpha_{\ell,j} \\ &\le 5.\end{aligned}$$

This completes the proof. ∎

We now describe the space. Consider the class $\mathcal{F}$ of functions that can be written $f = \sum_{r \ge 1} \alpha_r f_r$ where $\alpha_r \ge 0$, $\sum_r \alpha_r = 1$, $f_r \in \mathcal{F}_r$. Then for a sequence $x = (x(i))_{i \ge 1}$, we set

$$\|x\| = \sup_{f \in \mathcal{F}} \sum_{i \ge 1} |x(i)| \sqrt{f(i)}.$$

This obviously defines a symmetric sequence space. To simplify notation, for two functions x, y on $\mathbb{N}$, we write

$$\langle x, y \rangle = \sum_{i \ge 1} x(i) y(i)$$

so that

$$\|x\| = \sup_{f \in \mathcal{F}} \langle |x|, \sqrt{f} \rangle.$$

3. Failure of cotype

Since X satisfies the Orlicz property (as will be shown in Section 4) it is of cotype q for each $q < 2$. It is then known that for each sequence $(x_j)_{j\le N}$ of X one has

$$\mathop{Av}_{\eta_j=\pm 1} \|\sum_{j\le N} \eta_j x_j\| \le K\|z\|$$

where $z(i)^2 = \sum_{j\le N} x_j^2(i)$ for all $i \in \mathbb{N}$. (A more self-contained argument is given in [T].) It thus suffices to find vectors x_j for which the ratio $\|z\|^{-1}(\sum_{j\le N}\|x_j\|^2)^{1/2}$ is arbitrarily large. We consider an integer $\tau > 0$, that is fixed once and for all. We consider the vector $x = (x(i))$, where

$$x(i) = \frac{2^{-p/2}}{\sqrt{k_p}} \quad \text{if } k_{p-1} \le i < k_p, \quad p \le \tau$$

and $x(i) = 0$ if $i \ge k_\tau$.

We observe that $h \in \mathcal{F}_1$, so that $h \in \mathcal{F}$. Thus

$$\|x\| \ge \langle x, \sqrt{h}\rangle = \sum_{p\le\tau} \frac{k_p - k_{p-1}}{k_p} \ge \tau/2$$

since obviously $k_p \ge 2k_{p-1}$.

Consider now the vector $y = (y(i))$, where

$$y(i) = \frac{2^{-p/2}}{\sqrt{n_p}} \quad \text{for } n_{p-1} \le i < n_p, \quad p \le \tau$$

and $y(i) = 0$ if $i \ge n_\tau$. We observe that

$$\sum_{k_{p-1}\le i<k_p} x^2(i) \le 2^{-p} \le 2\Big(\sum_{n_{p-1}\le i<n_p} y^2(i)\Big)$$

since obviously $n_{p-1} \le n_p/2$. Thus, since obviously $n_p - n_{p-1} \ge k_p$, there exists a family $(x_j)_{j\le N}$ of vectors of X, such that x_j is of the type $x \circ \sigma_j$ for a certain permutation σ_j of $\mathbb{N}$, and such that

$$\frac{1}{N}\sum_{j\le N} x_j^2(i) \le 2y^2(i)$$

for all $i \in \mathbb{N}$. Since $\|x_j\| = \|x\|$ for all j, we have

$$\sum_{j\le N} \|x_j\|^2 = N\|x\|^2 \ge N\tau^2/4$$

and

$$\left(\sum_{j\le N}\|x_j\|^2\right)^{1/2} \ge \tau\sqrt{N}/2.$$

Thus, to conclude the proof it suffices, since τ is arbitrary, to show that $\|y_i\| \le 16 + 2\sqrt{\tau}$. To do this, given a function z with $z^2 \in \mathcal{F}$, we estimate $\langle |y|, |z|\rangle$. By definition of $\mathcal{F}$ we can write $z^2 = \sum_{r\ge 1}\alpha_r f_r$ with $f_r \in \mathcal{F}_r$, $\sum_{r\ge 1}\alpha_r = 1$, $\alpha_r \ge 0$. Obviously, we have, using the inequality $\sqrt{A+B+C} \le \sqrt{A}+\sqrt{B}+\sqrt{C}$, that

$$\sum_{i\ge 1}|y(i)||x(i)| \le \sum_{p\le\tau} I(p) + II(p) + III(p)$$

where

$$I(p) = \frac{2^{-p/2}}{\sqrt{n_p}}\sum_{n_{p-1}\le i<n_p}\sqrt{\sum_{r\le p}\alpha_r f_r(i)},$$

$$II(p) = \frac{2^{-p/2}}{\sqrt{n_p}}\sum_{n_{p-1}\le i<n_p}\sqrt{\alpha_{p+1}f_{p+1}(i)},$$

$$III(p) = \frac{2^{-p/2}}{\sqrt{n_p}}\sum_{n_{p-1}\le i<n_p}\sqrt{\sum_{r\ge p+2}\alpha_r f_r(i)}.$$

To handle $II(p)$, we observe that by Cauchy-Schwarz we have, by (2.7),

$$\begin{aligned} II(p) &\le 2^{-p/2}\sqrt{\alpha_{p+1}}\sqrt{\sum_{i\le n_p} f_{p+1}(i)} \\ &\le 2^{-p/2}\sqrt{\alpha_{p+1}}\sqrt{\sum_{i\le n_p} h(i)} \\ &\le 2\sqrt{\alpha_{p+1}}. \end{aligned}$$

Thus, by Cauchy-Schwarz we have

$$\sum_{p\le\tau} II(p) \le 2\sqrt{\tau}\sqrt{\sum_{p\le\tau}\alpha_{p+1}} \le 2\sqrt{\tau}. \tag{3.1}$$

To handle $III(p)$, we use Cauchy-Schwarz to see that

$$\begin{aligned} III(p) &\le 2^{-p/2}\sqrt{\sum_{i\le n_p}\sum_{r\ge p+2}\alpha_r f_r(i)} \\ &\le 2^{-p/2} \end{aligned}$$

since $\|f_r\|_\infty \le n_{r-2}^{-1} \le n_p^{-1}$ for $r \ge p+2$. To handle $I(p)$, using Lemma 2.3, we write

$$\sum_{r\le p} \alpha_r f_r = f' + f''$$

where the support of f' has cardinal $\le \sum_{r\le p} n'_r \le 2n'_p$ and where $\sum_{i\le n_p} f''(i) \le 5$. Now $I(p) \le IV(p) + V(p)$, where

$$IV(p) = \frac{2^{-p/2}}{\sqrt{n_p}} \sum_{n_{p-1}\le i<n_p} \sqrt{f'(i)},$$

$$V(p) = \frac{2^{-p/2}}{\sqrt{n_p}} \sum_{n_{p-1}\le i<n_p} \sqrt{f''(i)}.$$

By Cauchy-Schwarz,

$$\begin{aligned} IV(p) &\le \frac{2^{-p/2}}{\sqrt{n_p}}(\text{card supp}\, f')^{1/2}\Big(\sum_{i\le n_p} f'(i)\Big)^{1/2} \\ &\le 2^{-p/2}\left(\frac{2n'_p}{n_p}\right)^{1/2}(2^{p+2})^{1/2} \\ &\le 2^{-p/2} \end{aligned}$$

and

$$V(p) \le 2^{-p/2}\left(\sum_{i\le n_p} f''(i)\right)^{1/2} \le 3\cdot 2^{-p/2}.$$

It follows from these relations that

$$\langle |y|, |z|\rangle \le 5\sum_{p\ge 1} 2^{-p/2} + 2\sqrt{\tau}. \qquad \blacksquare$$

4. The Orlicz property

We consider vectors $(x_\ell)_{\ell\le N}$ of X, and we assume that

$$\forall\, \eta_\ell = \pm 1, \quad \|\sum_{\ell\le N} \eta_\ell x_\ell\| \le 1. \tag{4.1}$$

We set $S^2 = \sum_{\ell\le N} \|x_\ell\|^2$. Our aim is to show that this quantity is bounded by a constant.

There is no loss of generality to assume that the sequence $\|x_\ell\|$ decreases, so that

$$\|x_\ell\|^2 \leq \frac{S^2}{\ell}. \tag{4.2}$$

We set $\beta_\ell = \|x_\ell\|^2/S^2$, so that $\sum_{\ell \leq N} \beta_\ell = 1$. For each ℓ, we consider a function $g_\ell \in \mathcal{F}$ such that

$$\langle |x_\ell|, \sqrt{g_\ell}\rangle \geq 3\|x_\ell\|/4. \tag{4.3}$$

By definition, we can write

$$g_\ell = \sum_{r \geq 1} \alpha_{\ell,r} f_{\ell,r}$$

where $f_{\ell,r} \in \mathcal{F}_r$, $\sum\limits_r \alpha_{\ell,r} = 1$, $\alpha_{\ell,r} \geq 0$. We set

$$g'_r = \sum_{\ell \leq m_r} \beta_\ell \alpha_{\ell,r} f_{\ell,r} \quad \text{and} \quad \gamma_r = \sum_{\ell \leq m_r} \beta_\ell \alpha_{\ell,r}.$$

We observe that $\sum_r \gamma_r \leq 1$, and that, by Lemma 2.1 we have $g'_r \in 2\gamma_r \mathcal{F}_r$.

By Lemma 2.2, we can find a set A_r, with $\operatorname{card} A_r \leq rk_r$, such that the function

$$g''_r = 1_{A_r^c} \sum_{m_r < \ell \leq m_{r+1}, s < r} \beta_\ell \alpha_{\ell,s} f_{\ell,s}$$

belongs to $2\delta_r \mathcal{F}_{r+1}$, where

$$\delta_r = \sum_{m_r < \ell \leq m_{r+1}, s < r} \beta_\ell \alpha_{\ell,s}.$$

Observe that $\sum_r \delta_r \leq 1$. Thus, the function

$$g = \sum_{r \geq 1} g'_r + g''_r$$

belongs to $8\mathcal{F}$.

LEMMA 4.1: $\sum_\ell \|x_\ell 1_{A_r}\| \leq rk_r$.

Proof: We observe that $\|f\|_\infty \leq 1$ for $f \in \mathcal{F}_r$. Also, $\sup_{f \in \mathcal{F}} f(i) = 1$ for all i. By (4.1), we have

$$\forall i, \quad \forall\, \eta_\ell = \pm 1, \quad |\textstyle\sum_{\ell \leq N} \eta_\ell x_\ell(i)| \leq 1$$

so that $\sum_{\ell \le N} |x_\ell(i)| \le 1$. Thus,

$$\sum_{i \in A_r, \ell \le N} |x_\ell(i)| \le \operatorname{card} A_r \le r k_r.$$

Now

$$\|x_\ell 1_{A_r}\| \le \sum_{i \in A_r} |x_\ell(i)|$$

since $f \le 1$ for $f \in \mathcal{F}$. ∎

We set

$$H_r = \{\ell; \ell > m_r, \|x_\ell 1_{A_r}\| \ge 2^{-r-1}\|x_\ell\|\}$$

and $H = \bigcup_{r \ge 1} H_r$.

LEMMA 4.2: $\sum_{\ell \in H} \|x_\ell\|^2 \le S$.

Proof: By Lemma 4.1 and the definition of H_r, we get

$$\sum_{\ell \in H_r} \|x_\ell\| \le r 2^{r+1} k_r.$$

By (4.2), since $\ell \ge m_r$ for $\ell \in H_r$, we have

$$\sum_{\ell \in H_r} \|x_\ell\|^2 \le \max_{\ell \in H_r} \|x_\ell\| (\sum_{\ell \in H_r} \|x_\ell\|) \le \frac{S}{\sqrt{m_r}} (r 2^{r+1} k_r) \le S r 2^{-r}$$

by (2.5). Summation over r concludes the proof. ∎

Having controlled $\|x_\ell\|$ for $\ell \in H$ we try to control the other values of ℓ. We set

$$B_\ell = \bigcup \{A_s; m_s \le \ell\}.$$

If $\ell \notin H$, for $m_s \le \ell$, we have

$$\|x_\ell 1_{A_s}\| \le 2^{-s-1}\|x_\ell\|$$

so that

$$\|x_\ell 1_{B_\ell}\| \le \frac{1}{2}\|x_\ell\|.$$

Setting $y_\ell = x_\ell 1_{B_\ell^c}$, we have

$$\|y_\ell - x_\ell\| = \|x_\ell 1_{B_\ell}\| \leq \frac{1}{2}\|x_\ell\|.$$

Consider the function

$$f'_\ell = \sum_r \alpha_{\ell,r} f_{\ell,r} h_{\ell,r}$$

where $h_{\ell,r} = 1_{A_r^c}$ if $m_r \leq \ell$, and $h_{\ell,r} = 1$ if $m_r > \ell$. Then we have

$$\langle |x_\ell|, \sqrt{f'_\ell}\rangle = \sum_{i\geq 1} |x_\ell(i)|\ |f'_\ell(i)|^{1/2} \geq \sum_{i\geq 1} |y_\ell(i)||f_\ell(i)|^{1/2}$$

since $f'_\ell(i) = f_\ell(i)$ where $y_\ell(i) \neq 0$. Now, since $\sqrt{f_\ell} \in \mathcal{F}$,

$$\begin{aligned}\langle |y_\ell|, \sqrt{f_\ell}\rangle &\geq \langle |x_\ell|, \sqrt{f_\ell}\rangle - \langle |x_\ell - y_\ell|, \sqrt{f_\ell}\rangle \\ &\geq \langle |x_\ell|, \sqrt{f_\ell}\rangle - \|x_\ell - y_\ell\| \\ &\geq 3\|x_\ell\|/4 - \|x_\ell\|/2 \geq \|x_\ell\|/4.\end{aligned}$$

We set

$$a_\ell = \langle |x_\ell|, \sqrt{f'_\ell}\rangle.$$

We have shown that, for $\ell \notin H$,

$$a_\ell \geq \|x_\ell\|/4. \tag{4.4}$$

Now, setting $g' = \sum_{\ell\leq N} \beta_\ell f'_\ell$, we have

$$\begin{aligned}\sum_{\ell\leq N} a_\ell^2 \leq \sum_{\ell\leq N} a_\ell\|x_\ell\| &= \sum_{\ell\leq N} \|x_\ell\|\langle |x_\ell|, \sqrt{f'_\ell}\rangle \\ &= \sum_{\ell\leq N}\langle |x_\ell|, \|x_\ell\|\sqrt{f'_\ell}\rangle \\ &= S\sum_{\ell\leq N}\langle |x_\ell|, \sqrt{\beta_\ell f'_\ell}\rangle \\ &= S\sum_{i\geq 1}\left(\sum_{\ell\leq N} |x_\ell(i)|\sqrt{\beta_\ell f'_\ell(i)}\right) \\ &\leq S\sum_{i\geq 1}\left(\sum_{\ell\leq N} x_\ell^2(i)\right)^{1/2}\left(\sum_{\ell\leq N}\beta_\ell f'_\ell(i)\right)^{1/2} \\ &= S\left\langle \left(\sum_{\ell\leq N} x_\ell^2\right)^{1/2}, \sqrt{g'}\right\rangle\end{aligned} \tag{4.5}$$

where we have used Cauchy-Schwarz. Now, by Kintchine's inequality, we have, for some numerical constant $K(=\sqrt{2})$, that

$$\left(\sum x_\ell^2(i)\right)^{1/2} \leq K\, Av\, |\sum \eta_\ell x_\ell|.$$

The last term in (4.5) is thus less than

$$Av\, KS \sum_{i\geq 1} |\sum \eta_\ell x_\ell(i)|\, |g'(i)|^{1/2} \leq KS\, Av\, \langle |\sum \eta_\ell x_\ell|, \sqrt{g'}\rangle.$$

But we observe the crucial fact that $g' = g = \sum_{\ell\leq N} g'_\ell + g''_\ell \in 8\mathcal{F}$, so that this last term is at most

$$4KS\, Av\, \|\sum \eta_\ell x_\ell\| \leq 4KS$$

since the average is less than the max. By (4.5), we have

$$\sum_{\ell\notin H} \|x_\ell\|^2 \leq 64KS$$

and, combining with Lemma 4.2, we get

$$S^2 = \sum_{\ell\leq n} \|x_\ell\|^2 = \sum_{\ell\notin H} + \sum_{\ell\in H} \leq 65KS$$

so that $S \leq 65K$. ∎

References

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